Surface reconstruction from gradient data

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In many optical applications, surfaces have to be reconstructed from measured gradient data. For this purpose we use radial basis functions which are able to describe smooth surfaces on non-uniform measurement grids. Two approaches are compared for regular and irregular grid data with respect to their error tolerance and their computational effort.

1 Introduction

For measurement points (x_i, y_i), $i = 1,...,N$, and discrete gradient data $f_x(x_i, y_i)$, $f_y(x_i, y_i)$ the goal is to find a reconstruction Pf for the underlying surface f from the data. To solve the integration problem, classical methods use e.g. algebraic methods or Fourier methods that are restricted to regular grid data. In [2], the authors presented an alternative procedure which is based on radial basis function (RBF) interpolation. Recently, we have proposed a method using a RBF least square approximation [1].

We compare the interpolation and the least square method with respect to computational effort and error tolerance for regular and irregular spaced, noisy data.

2 Interpolation and least square approximation

For the interpolation approach of [2] it is required that the approximant Pf satisfies

$Pf_x(x_i, y_i) = f_x(x_i, y_i)$ and $Pf_x(x_i, y_i) = f_x(x_i, y_i)$ for $i = 1,...,N$, where the suffix denotes the partial derivatives in the coordinate directions. It is assumed that for a basis function Φ

$$Pf(x,y) = \sum_{j=1}^{N} \left(\alpha_j \Phi_x(x - x_j, y - y_j) + \alpha_{n+j} \Phi_y(x - x_j, y - y_j) \right).$$

For the least square approach in [1] it is required that the error of the gradient approximation is minimal, i.e.

$$\sum_{i=1}^{N} \left[\left(Pf_x(x_i, y_i) - f_x(x_i, y_i) \right)^2 + \left(Pf_y(x_i, y_i) - f_y(x_i, y_i) \right)^2 \right]$$

has to be minimized, where the approximant Pf is represented by $Pf(x,y) = \sum_{k=1}^{M} \beta_k \Phi(x - x_k, y - y_k)$ and $N \geq M$. Both approaches result in the solution of a linear system of equations for the coefficients.

3 Radial basis functions

In the approaches considered, special basis functions Φ are used which do not depend on the exact position of the measure points but only on their distance. Such functions are known as *radial basis functions*. A detailed description of such functions can be found e.g. in [3].

In our investigations we had a preference for the *inverse multiquadrics*

$$\Phi(x - x_j, y - y_j) = \left(\left((x - x_j)/c_x \right)^2 + \left((y - y_j)/c_y \right)^2 + 1 \right)^{-1/2}$$

and the *Wendland function* W_{42} which is defined by

$$\Phi(x - x_j, y - y_j) = (1 - r)_+^6 \left(\frac{35}{3} r^2 + 6r + 1 \right)$$

with $r = \sqrt{\left((x - x_j)/c_x \right)^2 + \left((y - y_j)/c_y \right)^2}$.

as a basis function. The choice of the basis function and the scaling factors c_x, c_y is important for the quality of the approximation. For the scaling factors it turned out that the region of interest should be taken into account by $c_x = \frac{w}{n-1} \cdot xscale$ and $c_y = \frac{h}{n-1} \cdot yscale$, where w and h are the width and height of this region and n is the number of measure points. For the inverse multiquadrics a value of 4-20 for $xscale$ and $yscale$ and in case of the Wendland function a value of 20-1000 for these parameters turned out to be optimal.

4 Test problem

For test purposes we chose a spherical cap and used this to compare the approaches of section 2 for regular and irregular grid points. We considered gradient data for this function, first without noise and later with noise in the order of 25 arcseconds, i.e. 0.1% in the gradient data.

5 Computational effort

We compared the two approaches with respect to the time which was needed for: 1. building up the system of equations (see Figure 1), 2. the solution of this system of equations (see Figure 2) and 3. the evaluation of the approximant Pf on a grid of 51 x 51 points. For the abscissa we took the number of measure points and for the ordinate we took the corresponding time of the different steps.

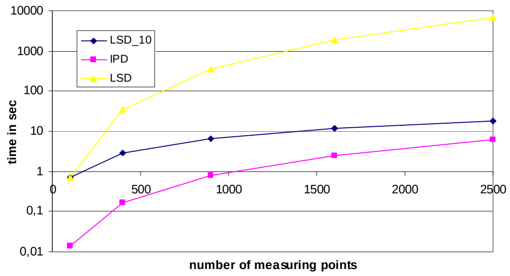


Fig. 1 *Time needed to build the systems of equations.*

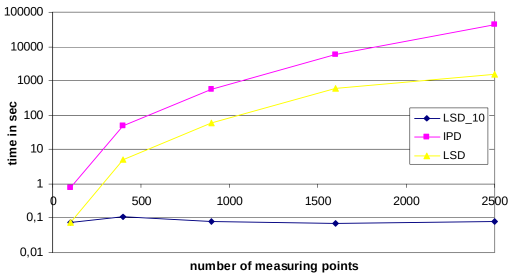


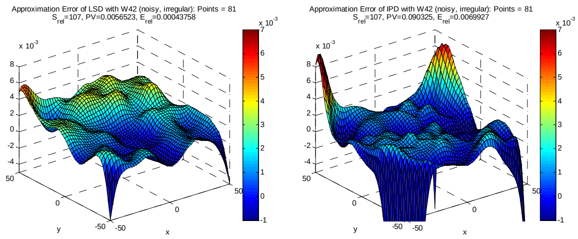
Fig. 2 *Time needed to solve the systems of equations.*

The building of the system of equations is more time consuming in the least square case than for the interpolation case. The bottleneck in the interpolation is the solution of the system of equations, where an additional factor of one hundred to one million seconds is needed, compared to the least square case (see Figure 2). To improve this problem, in [2] a decomposition technique of the region of interest is proposed. For the evaluation of the approximant, in both cases only about 10^{-3} seconds were needed.

6 Approximation error

For evaluation points (x_k, y_k) we compared the peek deviation $PV = \max_{(x_k,y_k)} \left| f(x_k, y_k) - Pf(x_k, y_k) \right|$ on a regular and on an irregular grid of data. In the second case we also added noise of 0.1% to the gradient data.

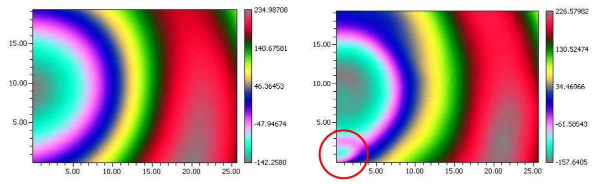
It turned out that for a regular grid data both approaches had a PV of order 10^{-3}, with the least square approximation showing an error of up to a factor of three better than the interpolation approach. For noisy data the difference was even larger. Here we found a PV in the order of 10^{-3} for the least square case compared to an error of 10^{-2} in the case of interpolation. The reason for this difference is that the interpolant shows larger errors at the boundary of the region of interest. Inside this region the interpolation seems to have comparable or even smaller errors than the least square approach (see Figure 4).



Least square error *Interpolation error*

Fig. 3 *PV for least square and interpolation for irregular, noisy gradient data.*

We also tested the two approaches for more realistic data. Figure 4 shows the gradient error for noisy gradient data gained from measurements of a Hartmann-Shack sensor.



Least square error *Interpolation error*

Fig. 4 *Least square error and interpolation error*

In the left lower corner of the right figure an error in the gradient data is clearly visible when using the interpolation approach. In contrast to this, in the least square approach this error is not visible any more, since this approach has a smoothening effect on errors.

To summarize: Both methods are well suited for the reconstruction of surfaces from gradient data, both avoid explicit integration. Nevertheless, from our experience, we favor the least square approach because of its better error tolerance and its lower computational effort.

References

[1] K.-H. Brenner, “Shifted base functions: An efficient and versatile new tool in optics”, to appear in proceedings of *WIO 2008*, Annecy, IOP Publishing 2008.

[2] S. Ettl, J. Kaminski, G. Häusler, “Generalized Hermite interpolation with radial basis functions considering only gradient data”, in *Curves and Surfaces*, Avignon 2006, Nashboro Press (Brentwood), pp. 141-149.

[3] G.E. Fasshauer, *Meshless approximation methods with Matlab*, Interdisciplinary Mathematical Sciences Vol. 6, (World Scientific, Singapore 2007).