Comparative study of computational methods for wavefront fitting

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Zernike polynomials one of the major tools in optical design application ranging from modeling optical surfaces to representing wavefront. We present comparative analysis of cubic B-spline model as an alternative to classical Zernike polynomials representation, and compare the efficacy of each represetation over a set of different sampling grids, when applied to complex wavefront fitting. The result shows that effectiveness of fit using cubic B-spline is better as compared to traditional Zernike polynomials. We also implement cubic B-spline fitting for efficiet tracing of light fields by geometrical optics.

1 Introduction

Numerical ray tracing and wavefront sensing technique are used to get the sampled measurements of wavefront. In the first case, we are always end up with the grid less measurements. The reconstruction of the wavefront from grid less measured points is a motivation of this study. The model Zernike polynomials(ZP) representation has been commonly chosen for wavefront fitting purpose. The objective of this work is to analyze the tensor product(TP) B-spline fitting method as an alternative to the Zernike technique when complex wavefront are involved.

We consider the Frank's test function as a complex wavefront and sampled it on equidistant, grid less and edge-clustered grids See Fig. 1 for an example. Simulations are compared the quality of fitted representations using ZP and TP B-spline methods. An analysis of the efficacy of computation ZP coefficients and control points of TP B-spline basis function is also compared.

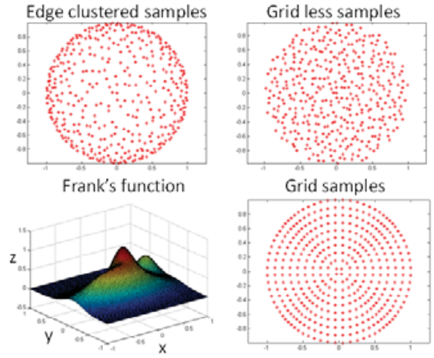

Fig. 1 *The sampling grids types.*

2 Fitting Techniques

The short mathematical analysis of the two fitting techniques analyzed will enable us to introduce the wavefront reconstruction principles on which they operate. The major theoretical concept involved in both methods are presented next.

2.1 Zernike

Zernike polynomial are complete and orthogonal basis over the unit circle. The ZP are defined in standard form in Born and Wolf [1]. A wavefront over a circular aperture may be represented as a function $\mathbf{W}(\rho,\theta)$ that is a weighted sum of ZP basis functions as

$$\mathbf{W}(\rho,\theta)=\sum_{k=1}^{N} c_k Z_n^m(\rho,\theta), \tag{1}$$

where c_k represents the coefficient associated with each ZP.

There are two major bottlenecks in representing complex wavefront with ZP. One is the numerical ill-conditioning associated with computation of the higher-order ZP that might be required to represent wavefront. Forbes addressed this bottleneck with the development of three-term recurrence relations for ZP that were described in [2]. Second is the substantial number of Zernike terms required, sometimes thousands. The rate of convergence to an adequate number of terms in the representation is sensitive to the wavefront sampling [3].

Eq. (1) shows the global fitting principle of the ZP representation; that is, the data $\mathbf{W}(\rho,\theta)$ are approximated by a polynomial function of degree k extended over the whole domain. Given L discrete data point from the measured wavefront $\mathbf{W_l}(\rho_l,\theta_l)$ $l=1,\cdots,L$, where (ρ_l,θ_l) are the normalized 2D polar coordinates, the only unknown parameter are the ZP coefficients c_k. The matrix form of Eq. (1) is shown as

$$\mathbf{W}=\mathbf{A}\mathbf{c}. \tag{2}$$

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The matrix $\mathbf{A}$ is $L \times N$, where N is the number of ZP coefficients to be fit, and L is the number of sample points, and $L > N$.

The general least squares that may be unstable, a $\mathbf{QR}$ decomposition of $\mathbf{A}$ is taken leading to the coefficients

$$\mathbf{c} = \mathbf{R}^{-1}(\mathbf{Q}^{*}\mathbf{f}). \qquad (3)$$

The $\mathbf{QR}$ based algorithm given in Eq. (3) to compute the solution of the least squares system is very stable.

2.2 B-spline

B-spline is a piecewise polynomial functions, which are used for complex free-form smooth surface generation, for detail see [4]. The TP B-spline model wavefront representation may be written as

$$\mathbf{W}(x, y) = \sum_{i=0}^{N}\sum_{j=0}^{M} \mathbf{N}_{i,p}(x)\mathbf{N}_{j,q}(y)\mathbf{P}_{i,j} \qquad (4)$$

where $\mathbf{P}_{i,j}$ are the control points and $\mathbf{N}_{i,p}(x)$ and $\mathbf{N}_{j,q}(y)$ are ith and jth B-spline basis functions, defined over the knot vectors $\mathbf{U} = \{u_{s,u-2}, \cdot_{n+2}\}$ and $\mathbf{V} = \{v_{-2}, \cdots, v_{m+2}\}$, of degree p and q in the x and y direction respectively. For practical purposes, it is convenient to reshuffle the two-dimensional index into a one-dimensional index. Then we have

$$\mathbf{W}(x, y) = \sum_{k=0}^{K} \mathbf{P}_k \mathbf{B}_k(x, y) \qquad (5)$$

where $\mathbf{B}_{j\times(N+1)+i} = \mathbf{N}_i(u)\mathbf{N}_j(v)$.

In case of univariate fitting with B-spline curves there exists a unique characterization of the solvability of fitting problem based on the location of the data point w.r.t the knots [5].

In the bivariate case no such unique characterization exists. Trying to find appropriate knot vectors for the parameter space is a very difficult task and is most likely to fail. Furthermore, for grid less sampled data, the method discussed in [4] to find knot vectors cannot be applied here.

We define parameter area of TP B-spline wavefront as the bounding box $[u_{min}, u_{max}]$x$[v_{min}, v_{max}]$ of all measured points. Determine uniform knot vector U and V with knot spacing $h_u = u_{i+1} - u_i$ and $h_v = v_{i+1} - v_{i+1}$. Determine the grid such that all data points (x_l, y_l) lie in the inner region $[u_1, u_{N-1}]$x$[v_1, v_{M-1}]$ of the parameter region, i.e.: $u_{min} = u_1$, $u_{max} = u_{N-1}$ and $v_{min} = v_1$, $v_{max} = v_{M-1}$.

In Eq. (5) a least squares system of equation is shown as

$$\mathbf{W} = \mathbf{BP}. \qquad (6)$$

The matrix $\mathbf{B}$ is $L \times (K + 1)$, where $(K + 1)$ is the number of TP B-spline basis functions to be fit, and L is the number of sample points, and $L > (K + 1)$.

Then, the $(K + 1) = (N + 1) \times (M + 1)$ control points $\mathbf{P}_k$ can be determined by solving the following $(K + 1) \times (K + 1)$ linear system

$$(\mathbf{B^T B})\mathbf{P} = \mathbf{B^T W}. \qquad (7)$$

3 Simulation Results

To study the capabilities of the TP B-spline and ZP fitting methods when they are applied to complex wavefront, three different grid type of 4096 sampled points of theoretical wavefront were generated. For evaluating the ZP and TP B-spline representation, ZP of order 40 and control points grid 40×40, respectively, were considered for the fitting. The comparison is accomplished using the rms fit deviation, which gives the rms difference values between the measured and reconstructed wavefronts.

Method	edge	grid less	gridded	time(sec)
ZP	$4.1e^{-05}$	3.2^{-05}	3.6^{-05}	0.97
B-spline	2.7^{-10}	3.6^{-10}	4.4^{-10}	0.39

Tab. 1 *RMS of measured and reconstructed wavefronts.*

The Tab. 1 contain rms fitting errors of three types of grids and the computational time of edge-clustered grid for both fitting methods.

4 Conclusion

The results obtained shows that B-spline polynomials fitting method is efficient as well as have better performance. They are well suited as an alternative to the Zernike polynomials. Furthermore, B-spline have the advantage of locally defined and having great flexibility that allow their smoothness and polynomial degree to be controlled.

They are useful in freeform surfaces manufacturing as far as computerized numerically controlled (CNC) grinding and polishing tools are easily commanded using them. The tested wavefront is not usually described on the same basis when it is measured and when it is used to control the polishing machine, which would obviously be the optimum scheme for industrial production.

References

[1] M. Bom and E. Wolf, "Principles of optics," in *Pergamon Oxford (Ed.)* (1980).

[2] G. Forbes, "Robust and fast computation for the polynomials of optics," Optics express **18**(13), 13,851–13,862 (2010).

[3] I. Kaya, K. P. Thompson, and J. P. Rolland, "Edge clustered fitting grids for φ-polynomial characterization of freeform optical surfaces," Optics express **19**(27), 26,962–26,974 (2011).

[4] L. Piegl and W. Tiller, "The NURBS book. 1997," Monographs in Visual Communication (1997).

[5] C. De Boor, *A practical guide to splines; rev. ed.*, Applied mathematical sciences (Springer, Berlin, 2001). URL `https://cds.cern.ch/record/1428148`.