The Geometric Fourier Transform

Frank Wyrowski* and Christian Hellmann**
*Applied Computational Optics Group, Institut für Angewandte Physik, Friedrich-Schiller-Universität Jena
**Wyrowski Photonics UG
mailto:frank.wyrowski@uni-jena.de

The Fourier transform of the electromagnetic field components at different planes of a system is a frequent operation in physical optics modeling which connects the space and k domains. We introduce the so-called geometric zone of a field, in which the Fourier transform can be obtained without integration and, in conclusion, in a very numerically efficient manner. In the geometric field zone the field is governed by its wavefront phase, thus allowing us to apply the concept of stationary phase to the Fourier transform integral. We refer to the resulting Fourier transform algorithm as the geometric Fourier transform: a technology which turns out to be a fundamental pillar for fast physical optics.

1 Fourier transform in optics

In physical optics we deal with the six complex field components of the electromagnetic field ($\boldsymbol{E}$ and $\boldsymbol{H}$ respectively). In the space domain they are denoted by

$$V_\ell(\boldsymbol{\rho}, z, \omega) = |V_\ell(\boldsymbol{\rho}, z, \omega)| \exp\big(\mathrm{i}\gamma_\ell(\boldsymbol{\rho}, z, \omega)\big) \quad (1)$$

with $\boldsymbol{\rho} = (x, y)$ and $\ell = 1, \ldots, 6$, e.g. $V_1 = E_x$ and $V_5 = H_y$. The Fourier transform into the k domain is defined by

$$\begin{aligned}\tilde{V}(\boldsymbol{\kappa}, z, \omega) &= \mathcal{F}_k V(\boldsymbol{\rho}, z, \omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\boldsymbol{\rho}, z, \omega) \exp(-\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) \, \mathrm{d}x \, \mathrm{d}y \,,\end{aligned} \quad (2)$$

where we use the notation

$$\tilde{V}_\ell(\boldsymbol{\kappa}, z, \omega) = |\tilde{V}_\ell(\boldsymbol{\kappa}, z, \omega)| \exp\big(\mathrm{i}\tilde{\gamma}_\ell(\boldsymbol{\kappa}, z, \omega)\big) \,. \quad (3)$$

The numerical evaluation of the integral in Eq. 2 requires sampling the fields in x and k domain. We denote the number of sampling points with N. The resulting discrete Fourier Transform constitutes an N^2 operation. However, the Fast Fourier Transform (FFT) algorithm is linear in N, which enables fast physical optics modeling in principle. However, the FFT requires the sampling of $\exp\big(\mathrm{i}\gamma_\ell(\boldsymbol{\rho}, z, \omega)\big) = \cos[\gamma_\ell(\boldsymbol{\rho}, z, \omega)] + \mathrm{i}\sin[\gamma_\ell(\boldsymbol{\rho}, z, \omega)]$. In optics we typically have phase functions with strong gradients, which leads to huge values of N. Just in very paraxial optics N can be small. Thus, though the FFT is linear in N, we easily encounter the problem in optics that N is too large for a fast computation of the Fourier transform. This is a serious obstruction for a fast physical optics concept.

For further investigation we decompose γ_ℓ into (skipping ω)

$$\gamma_\ell(\boldsymbol{\rho}, z) = \phi_\ell(\boldsymbol{\rho}, z) + \psi(\boldsymbol{\rho}, z) \quad (4)$$

with the wavefront phase ψ, which is the same for all components. Obviously, the decomposition in Eq. 4 is ambiguous and relies on a good strategy in phase treatment in modeling, starting with the source field. With the definition $U_\ell(\boldsymbol{\rho}, z) = |V_\ell(\boldsymbol{\rho}, z)| \exp(\mathrm{i}\phi_\ell(\boldsymbol{\rho}, z))$ the decomposition

$$V_\ell(\boldsymbol{\rho}, z) = U_\ell(\boldsymbol{\rho}, z) \exp\big(\mathrm{i}\psi(\boldsymbol{\rho}, z)\big) \quad (5)$$

results. Analogously we obtain

$$\tilde{V}_\ell(\boldsymbol{\kappa}, z) = \tilde{A}_\ell(\boldsymbol{\kappa}, z) \exp\big(\mathrm{i}\tilde{\phi}(\boldsymbol{\kappa}, z)\big) \,, \quad (6)$$

with the wavefront phase $\tilde{\phi}$ in the k-domain. It should be mentioned that a decomposition according to Eq. 5 with $U = |V|$ is known in geometrical optics and then $\psi(\boldsymbol{\rho}, z) = kS(\boldsymbol{\rho}, z)$ with the Eikonal S. We would like to emphasize, that the decomposition in Eq. 5 is more general and a pure mathematical approach in physical optics. Our goal can be stated as follows: We are interested in techniques to perform a Fourier transform without sampling the wavefront phase factors. When ψ and $\tilde{\phi}$ are in the form of quadratic polynomials that can be achieved by a semi-analytical Fourier transform [1]. Here we would like to discuss a concept which works for general wavefront phases but under the approximation of strong wavefront phases. It uses the concept of stationary phases.

2 Theory of geometric Fourier transform

The application of the stationary phase method is well known in optics, e.g. for the discussion of diffraction integrals in [2]. We apply it for a fast calculation of the Fourier transform integral of Eq. 2. To this end we assume that $\exp(\mathrm{i}\psi(\boldsymbol{\rho}, z) - \boldsymbol{\kappa} \cdot \boldsymbol{\rho})$ possesses in the plane through z much higher spatial frequencies than $U(\boldsymbol{\rho}, z)$ with the exception of the neighborhood

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of a critical point. According to the concept of stationary phase that leads directly to the basic equation (skipping the z)

$$\nabla_\perp \psi(\boldsymbol{\rho}) = \boldsymbol{\kappa}(\boldsymbol{\rho})\,, \tag{7}$$

with $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$. Eq. 7 states a mapping between $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$. We assume that this mapping is open, bijective and continuous, which means it constitutes a homeomorphism. This is a mathematical expression for the smoothness of the wavefront phase $\psi(\boldsymbol{\rho})$ and ensures that the resulting field in k can be interpolated on a non-equidistant grid. In optics this condition is typically satisfied when the field is not in a caustic zone. The stationary phase concept also reveals

$$\tilde{V}(\boldsymbol{\kappa}) \approx \mathcal{F}_k^{\text{geom}} V(\boldsymbol{\rho}) = \alpha(\boldsymbol{\kappa}) \tilde{A}(\boldsymbol{\kappa}) \exp\big(\mathrm{i}\tilde{\phi}(\boldsymbol{\kappa})\big) \tag{8}$$

with the Legendre transformation of $\psi(\boldsymbol{\rho})$ according to

$$\tilde{\phi}(\boldsymbol{\kappa}) = \psi\big(\boldsymbol{\rho}(\boldsymbol{\kappa})\big) - \boldsymbol{\kappa} \cdot \boldsymbol{\rho}(\boldsymbol{\kappa})\,, \tag{9}$$

the complex function

$$\tilde{A}(\boldsymbol{\kappa}) = U\big(\boldsymbol{\rho}(\boldsymbol{\kappa})\big) \tag{10}$$

and the weighting factor $\alpha(\boldsymbol{\kappa})$ which depends on the second derivatives of $\psi(\boldsymbol{\rho})$. The result expresses the Fourier transform by a mapping of the field values in the space domain into the k domain with an additional weight factor, which depends, as the mapping itself, on the wavefront phase only. Thus, the Fourier transform performs mainly a geometric distortion of the field distribution and we refer to it as the geometric Fourier transform.

We have developed a numerical algorithm to perform a geometric Fourier transform. It uses a hybrid sampling of the field. The wavefront phase $\psi(\boldsymbol{\rho})$ itself can be parametrized, in contrast to the function $\exp(\mathrm{i}\psi(\boldsymbol{\rho}))$, by a small number $N(\psi)$ of non-equidistantly distributed values. The control points of a spline interpolation are possible candidates. Moreover, we must handle the sampling of the function $U(\boldsymbol{\rho})$ with $N(U)$ equidistantly distributed sampling points. In general we have $N(\psi) \ll N(U) \ll N(V)$. The numerics of the geometric Fourier transform is mainly based on linear operations in $N(\psi)$ and is therefore very fast. The smart inclusion of the $N(U)$ values of U can be done fast as well. The sampling of V can be completely avoided. In summary, the resulting numerical algorithm enables a very fast Fourier transform when the geometric Fourier transform is accurate enough, which is the case for strong wavefront phases. For weaker ones the semi-analytical Fourier transform is suitable and fast as well [1]. Together with the regular FFT, which is numerically the most efficient one for very weak wavefront phases, we obtain a powerful triad to handle all relevant Fourier transform situations. It is implemented in the second generation technology update of VirtualLab Fusion and constitutes the backbone for its fast physical optics technology [3]. As an example the Gouy phase shift is investigated with that concept [4].

3 Diffractive, geometric and far field zones

Let us consider a field in a plane z which can be transformed by the geometric Fourier transform with sufficient accuracy (specified by a quality criterion). Then we say that that plane is located in the geometric field zone (GFZ). Otherwise, the field is in its diffractive zone (DFZ). Naturally the diffractive field zone is located around the focal region whereas the GFZ appears for larger distances from the focal region. If the field is propagated further, the far field zone, which forms a subset of the geometric zone, is reached. In the geometric zone we do not restrict the wavefront phase ψ, that means we include also aberrations. If the geometric Fourier transform provides accurate results for a spherical ψ, the far field zone has been reached. This is summarized in Table 1. For diffraction-limited fields the geometric and the far field zones are identical. It should be emphasized that in each plane the zone characteristic of a field can be investigated by the geometric Fourier transform, which constitutes a purely mathematical concept. It turns out that physical optics modeling in the geometric zone of fields can be performed particularly fast, because numerically it deals mainly with the relatively small numbers $N(\psi)$ of wavefront phase samples.

Zone	**Geometric FT**	**Phase ψ**
diffractive zone	No	—
geometric zone	Yes	general
far zone	Yes	spherical

Tab. 1 *Definition of field zones.*

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