The Semi-Analytical Fast Fourier Transform

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We propose a way to handle the Fourier transform which does not require the sampling of second order polynomial phase terms, but rather treats them analytically. We present the theory alongside several examples to demonstrate its potential.

1 Introduction

Physical optics modeling requires frequent shifting from the space into the angular spectrum domain and vice versa. This is achieved by performing a Fourier transform of the electric and magnetic field components. Therefore, the Fast Fourier transform (FFT) algorithm constitutes the backbone for fast physical optics modeling [1]. The numerical effort of the FFT technique is approximately linear with the required number of sampling points of the complex amplitude of a field component. In optics we often deal with field components which possess a strong wavefront phase, e.g. a spherical one. However, due to the 2π modulo, complex sampling of the smooth wavefront phase leads to a huge numerical effort even in the case of the FFT.

2 Theory

2.1 Field Representation: Extract Quadratic Phase

We start from the notation in spatial domain. In this paper, we use the symbol V_ℓ with $\ell = 1, \ldots, 6$ for the six field components, that means $\boldsymbol{V} = (\boldsymbol{E}, \boldsymbol{H})$:

$$\begin{aligned} V_\ell(\boldsymbol{\rho}) &= U_\ell(\boldsymbol{\rho}) \exp\big(\mathrm{i}\psi(\boldsymbol{\rho})\big) \\ &= U_\ell(\boldsymbol{\rho}) \exp\big(\mathrm{i}\psi^{\mathsf{res}}(\boldsymbol{\rho})\big) \exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big) \\ &= U_\ell^{\mathsf{res}}(\boldsymbol{\rho}) \exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big). \end{aligned} \tag{1}$$

In Eq. 1, we assume that the field $V_\ell(\boldsymbol{\rho})$ consists of two parts, the diffractive field part $U_\ell(\boldsymbol{\rho})$ and a smooth wavefront phase part $\exp\big(\mathrm{i}\psi(\boldsymbol{\rho})\big)$. For the subsequent derivation, we extract the quadratic phase $\exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big)$ from the wavefront phase and consider the rest part as the residual field $U_\ell^{\mathsf{res}}(\boldsymbol{\rho})$. We assume that $\exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big)$ can be given by its real-valued coefficients C and $\boldsymbol{D} = (D_x, D_y)$:

$$\psi_{\mathsf{q}}(\boldsymbol{\rho}) = Cxy + D_x x^2 + D_y y^2. \tag{2}$$

Obviously, in case of a strong quadratic phase, the full field $V_\ell(\boldsymbol{\rho})$ would require much more sampling effort than the residual field. Hence, our aim is to calculate the Fourier transform of $V_\ell(\boldsymbol{\rho})$ via FFT and without the sampling of the quadratic phase term $\exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big)$.

2.2 Semi-Analytical Fourier Transform

From the convolution theorem, we know

$$\begin{aligned} \tilde{V}_\ell(\boldsymbol{\kappa}) &= \boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[V_\ell(\boldsymbol{\rho})] \\ &= \boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[U_\ell^{\mathsf{res}}(\boldsymbol{\rho})] * \boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[\exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big)]. \end{aligned} \tag{3}$$

In general, the term $\boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[U_\ell^{\mathsf{res}}(\boldsymbol{\rho})]$ must be treated numerically. On the other hand, from mathematics [2] we know

$$\iint \exp\big(-ax^2 + bx + c\big)\mathrm{d}x = \sqrt{\frac{\pi}{a}} \exp\big(\frac{b^2}{4a} + c\big) \tag{4}$$

which is valid for any complex $a, b, c, \in \mathbb{C}$, provided that $\Re\{a\} \geq 0$ and $a \neq 0$.

With the help of this mathematical tool, the analytical representation of the term $\boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[\exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big)]$ is deduced out:

$$\boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[\exp\big(\mathrm{i}\psi_{\mathsf{q}}(\boldsymbol{\rho})\big)] = \alpha \exp\big(\mathrm{i}\tilde{\phi}_{\mathsf{q}}(\boldsymbol{\kappa})\big) \tag{5}$$

with

$$\tilde{\phi}_{\mathsf{q}}(\boldsymbol{\kappa}) = \frac{C}{\gamma} k_x k_y + \frac{D_x}{\gamma} k_x^2 + \frac{D_y}{\gamma} k_y^2 \tag{6}$$

with constant factors $\alpha = \alpha(C, \boldsymbol{D})$ and here $\gamma = C^2 - 4D_x D_y$.

Plugging Eq. 5 into Eq. 3, by exchanging the order of convolution and Fourier transform integral, we find that $\tilde{V}_\ell(\boldsymbol{\kappa})$ can be expressed by

$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \tilde{A}_\ell^{\mathsf{res}}(\boldsymbol{\kappa}) \exp\big(\mathrm{i}\tilde{\phi}_{\mathsf{q}}(\boldsymbol{\kappa})\big) \tag{7}$$

with

$$\tilde{A}_\ell^{\mathsf{res}}(\boldsymbol{\kappa}) = \alpha \boldsymbol{\mathcal{F}}_{\boldsymbol{\beta}}^{-1}\left[\tilde{U}_\ell^{\mathsf{res}}(\boldsymbol{\kappa}) \exp\big(\mathrm{i}\tilde{\phi}_{\mathsf{q}}(\boldsymbol{\kappa})\big)\right]. \tag{8}$$

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Here, $\tilde{U}_\ell^{\text{res}}(\boldsymbol{\kappa}) = \boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}[U_\ell^{\text{res}}(\boldsymbol{\rho})]$ and the coordinate factor $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{\kappa}, C, \boldsymbol{D})$. Eq. 7-8 are the mathematical expressions of the semi-analytical Fourier transform. It indicates that the FFT of the full field can be replaced by two FFT of the residual field.

3 Numerical Simulations

The concepts were implemented in the physical optics modeling and design software Wyrowski VirtualLab Fusion [3].

3.1 Validity Test 1: Pure Quadratic Phase

In the first test group, we prepare a residual field $U_\ell^{\text{res}}(\boldsymbol{\rho})$ whose amplitude information is shown in Fig. 1, and zero phase. We multiply different quadratic phase terms $\exp\left(\mathrm{i}\psi_{\text{q}}(\boldsymbol{\rho})\right)$ on it to construct $V_\ell(\boldsymbol{\rho})$. Then we respectively apply FFT and semi-analytical FFT to the full field $V_\ell(\boldsymbol{\rho})$.

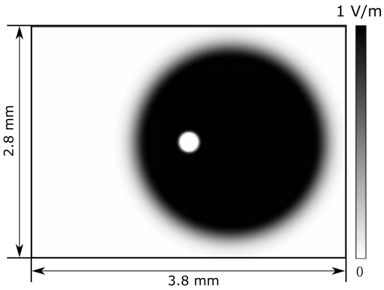


Fig. 1 *Amplitude (E_x-component) of $U_\ell^{\text{res}}(\boldsymbol{\rho})$: E_x-polarized, @532 nm*

Fig. 2 shows the required sampling points of the FFT and semi-analytical FFT in different cases. We can find that when the field possesses a strong quadratic phase, the semi-analytical FFT requires many fewer sampling points than FFT.

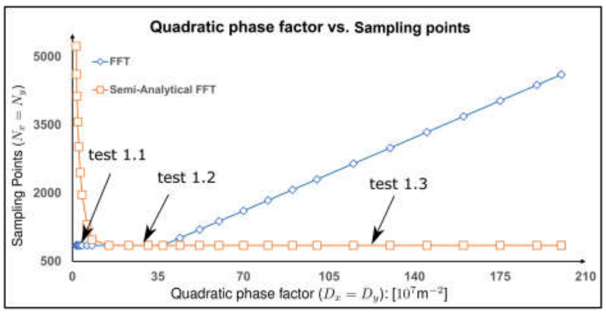
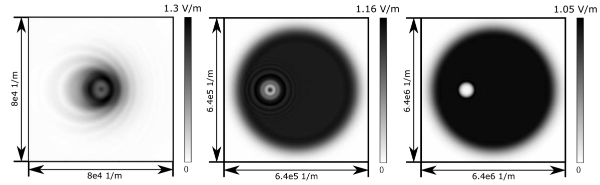


Fig. 2 *Sampling points (FFT and semi-analytical FT) vs. quadratic phase factor $\boldsymbol{D} = (D_x, D_y)$.*

In Fig. 3 we present the amplitude of angular spectra of three typical positions. The physical meaning of wavefront phase is revealed so that when the wavefront phase is very small, diffraction effects predominate in the FT. Otherwise, as the wavefront phase increases, the FT shows more and more geometric characteristics.



(a) Test 1.1 (b) Test 1.2 (c) Test 1.3

Fig. 3 *Amplitude (E_x-component) of $\tilde{V}_\ell(\boldsymbol{\kappa})$ to different quadratic phase factors.*

3.2 Validity Test 2: Spherical Phase

In the second group, we multiply another kind of phase on the prepared field: a spherical phase $\left(\psi(\boldsymbol{\rho}) = \exp\left(\mathrm{i}kr\sqrt{(\frac{x}{r})^2 + (\frac{y}{r})^2 + 1}\right)\right)$.

Unlike in test 1, we can't handle the whole spherical phase analytically but only the quadratic part of it. Therefore, the phase of the residual field is no longer zero but the difference between the spherical and the quadratic phase, and it would become larger and larger as the spherical radius r decreases.

The sampling points of the FFT and semi-analytical FFT in different cases are presented in Fig. 4. The result implied that in case of a strong spherical phase, due to the phase difference, $U_\ell^{\text{res}}(\boldsymbol{\rho})$ requires more sampling points which lead to the sampling effort of semi-analytical FT also increasing.

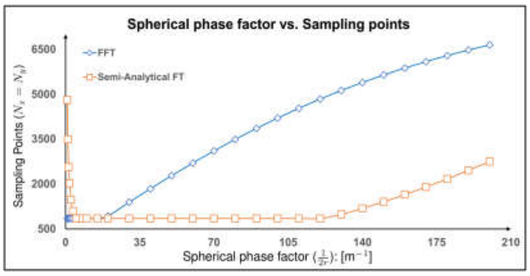


Fig. 4 *Sampling points (required in FFT and semi-analytical FFT) vs. spherical phase factor ($\frac{1}{2r}$).*

4 Conclusion

We demonstrate the derivation of the semi-analytical FFT and present several numerical examples. The facts show that sampling of the semi-analytical FFT only depends on the residual field. In case of the field with a strong wavefront phase, the semi-analytical FFT requires significantly fewer sampling points.

References

[1] E. O. Brigham, "The fast Fourier transform and its applications." (1988).

[2] L. Mandel and E. Wolf, *Optical coherence and quantum optics* (Cambridge university press, 1995).

[3] "Wyrowski VirtualLab Fusion, developed by Wyrowski Photonics UG, distributed by LightTrans GmbH," .